OBJECTIVE MATHEMATICS

Volume 1

Descriptive Test Series

Prof. M. L. Khanna
Bhushan Muley

CHAPTER-16 : THE ELLIPSE

UNIT TEST-1

1. If the chord of contact of the tangents drawn from the point (α,β) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2+y^2=c^2$, then the point (α,β) lies on the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$. T or F?

2. The condition that the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose middle point is (x_1, y_1) subtends a right angle at the centre of the ellipse is ....

3. The locus of the middle points of the chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touching the ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ is ....

4. The locus of the point the chord of contact of tangents from which to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ subtends a right angle at the centre of the ellipse is ....

5. A chord PQ of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ subtends a right angle at the centre of the ellipse. The locus of the point of intersection of the tangents to the ellipse at P and Q is ....

6. The locus of the point the chord of contact of tangents from which to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2+y^2=c^2$ is ....

7. The condition that the chord of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, whose middle point is (x_1, y_1) subtends a right angle at the centre of the ellipse is ....

8. The locus of the poles of the normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is ....

9. The locus of the mid-points of the lines joining the extremities of two semi-conjugate diameters of an ellipse is............

10. The locus of the point of intersection of tangents at the ends of semi-conjugate diameters of an ellipse is..........

11. Tangents are drawn from the points on the line $x-y-5=0$ to $x^2+4y^2=4$, then all the chords of contact pass through a fixed point, whose co-ordinates are ....

12. The length of the common chord of the ellipse $\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}=1$ and the circle $(x-1)^2+(y-2)^2=1$ is ....

13. Prove that, in an ellipse, the distance between the centre and any normal does not exceed the difference between the semi-axes of the ellipse.

HINTS AND SOLUTIONS

1. Chord of contact is $\frac{\alpha x}{b^2}+\frac{\beta y}{b^2}=1$

It touches the circle $x^2+y^2=c^2 \quad \therefore \quad p=r$

2. $\frac{x_1^2}{a^4}+\frac{y_1^2}{a^4}=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$

$T=S_1$

$\therefore \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$

Now make the equation of ellipse homogeneous with the help of above line and as it represents a pair of perpendicular lines.

$\therefore \quad A+B=0$ as in

3. $\frac{\alpha^2x^2}{a^4}+\frac{\beta^2y^2}{a^4}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$

$T=S_1$ is a tangent to $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$

$\therefore \quad A^2L^2+B^2M^2=N^2$

4. $\frac{x^2}{a^4}+\frac{y^2}{a^4}=\frac{1}{a^2}+\frac{1}{b^2}$

Let the point from which tangents are drawn be (x_1,y_1) so that the equation of chord of contact is

$$\frac{xx_1}{a^2}=\frac{yy_1}{b^2}=1 \qquad \ldots(1)$$

It subtends a right angle at the centre (0, 0) of ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Making the equation of the ellipse homogeneous by the help of equation of chord of contact we get the equation of the lines joining the origin to the points of intersection of the chord of contact and the ellipse as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)^2$$

or $x^2\left(\frac{1}{a^2}-\frac{x_1^2}{a^4}\right)-\frac{2xyx_1y_1}{a^2b^2}+y^2\left[\frac{1}{b^2}-\frac{y_1^2}{b^4}\right]=0 \qquad \ldots(2)$

Since these lines are at right angles therefore sum of the coefficients of x^2 and y^2 in (2) should be zero.

$$\therefore \quad \frac{1}{a^2}-\frac{x_1^2}{a^4}+\frac{1}{b^2}-\frac{y_1^2}{b^4}=0$$

Hence the locus of the point (x_1,y_1) is

$$\frac{x^2}{a^4}+\frac{y^2}{a^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

5. $\frac{x^2}{a^4}+\frac{y^2}{a^4}=\frac{1}{c^2}$

c.c. $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ of (x_1,y_1) touches the circle.

$x^2+y^2=c^2$. Apply $p=r$

6. $\frac{x^2}{81}+\frac{y^2}{16}=13$

7. $13\left(4x_1^2+9y_1^2\right)^2=36\left[16x_1^2+81y_1^2\right]$

8. $\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2$

Any normal to the ellipse is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \qquad \ldots(1)$$

If its pole be (h,k) then (1) is same as polar of (h,k) *i.e.*

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1$$

Comparing (1) and (2), we get

$$\frac{a}{\cos\theta}\cdot\frac{a^2}{h}=\frac{-b}{\sin\theta}\cdot\frac{b^2}{k}=\frac{a^2-b^2}{1}$$

$$\therefore \quad \frac{a^3}{h}=(a^2-b^2)\cos\theta, \frac{b^3}{k}=-(a^2-b^2)\sin\theta$$

Square and add

$\frac{a^6}{h^2}+\frac{b^6}{k^2}=(a^2-b^2)^2\left(\cos^2\theta+\sin^2\theta\right)=(a^2-b^2)^2$

9. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}.$

Let P be $\left(a\cos\theta, b\sin\theta\right)$ then $D(-a\sin\theta, b\cos\theta)$

If (h,k) be the mid-point of PD, then

$$2h=a\left(\cos\theta-\sin\theta\right), 2k=b\left(\sin\theta+\cos\theta\right)$$

$$\therefore \quad \frac{4h^2}{a^2}+\frac{4k^2}{b^2}=\left(\cos\theta-\sin\theta\right)^2+\left(\sin\theta+\cos\theta\right)^2=2$$

10. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$

Tangent at P is $\frac{x\cos\theta}{a}+\frac{y\cos\theta}{b}=1$

Tangent at D is $\frac{-x\sin\theta}{a}+\frac{y\cos\theta}{b}=1$

In order to find the locus of the point of intersection of these tangents we have to eliminate θ for which we square and add.

$$\therefore \quad \frac{x^2}{a^2}.1+\frac{y^2}{b^2}.1+0=2 \quad \Rightarrow \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=2$$

11. $(4/5, -1/5)$

Let (h, k) be any point on the line

$\therefore$ $x - y - 5 = 0$ $\quad\therefore\quad h - k = 5 \quad\therefore\quad h - 5 = k$

Its chord of contact w. r. t. $x^2 + 4y^2 = 4$ is

$xh + 4yk = 4$

or $xh + 4y(h-5) = 4$ or $xh + 4hy - 20y = 4$

or $h(x+4y) - 4(5y+1) = 0$

or $4(5y+1) - h(x+4y) = 0$

Above is of the form $P - \lambda Q = 0$ which always passes through the intersection of $P = 0$ and $Q = 0$ *i.e.* $(4/5, -1/5)$ which is fixed.

12. zero.

Eliminating $(y-2)^2$ between the two equations we get

$$\frac{(x-1)^2}{9} + \frac{1}{4}\left[1 - (x-1)^2\right] = 1$$

or $$\frac{(x-1)^2}{9} + \frac{1}{4} - \frac{(x-1)^2}{4} = 1$$

or $$(x-1)^2\left[\frac{1}{9} - \frac{1}{4}\right] = 1 - \frac{1}{4} = \frac{3}{4}$$

or $$(x-1)^2 = -27/5.$$

Above gives imaginary values of x and hence of y.

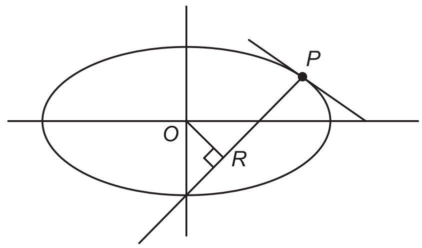
Thus the two curves do not intersect.

13. Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of normal at $P(\theta)$ is

$$(a\sec\theta)x - (b\,\text{cosec}\,\theta)y - a^2 + b^2 = 0$$

distance of normal from centre



$$= OR = \frac{|a^2 - b^2|}{\sqrt{a^2 + b^2 + (a\tan\theta)^2 + (b\cot\theta)^2}}$$

$$= \frac{|a^2 - b^2|}{\sqrt{(a+b)^2 + (a\tan\theta - b\cot\theta)^2}}$$

$\therefore$ $(a+b)^2 + (a\tan\theta - b\cot\theta)^2 \geq (a+b)^2$

or $\leq \frac{|a^2 - b^2|}{\sqrt{(a+b)^2}} \Rightarrow |OR| \leq (a - b)$